The failure distribution in percolation models of breakdown

1987 J. Phys. A: Math. Gen. 20 L411

(http://iopscience.iop.org/0305-4470/20/6/013)

J. Phys. A: Math. Gen. **20** (1987) L411-L415. Printed in the UK

LETTER TO THE EDITOR

The failure distribution in percolation models of breakdown

P M Duxbury† and P L Leath‡

† Department of Physics and Astronomy, Michigan State University, East Lansing, MI 48824-1116, USA

‡ Department of Physics and Astronomy, Rutgers University, Piscataway, NJ 08854, USA

Received 18 November 1986

Abstract. The probability of finding a *largest defect cluster* of size n in a percolation network is calculated analytically using a new distribution function scaling equation. From this result and the stress (voltage) enhancement at the tip of the most critical defect in the network, the probability of failure in percolation models of breakdown is calculated. For defect fractions less than the percolation point, this distribution is found to be of the form exponential of an exponential. Numerical simulations on the two-dimensional random fuse network confirm the new distribution function and convincingly distinguish between it and the Weibull form most often used in the fitting of breakdown data.

Defects are well known to be of dominating importance in determining both the electrical and mechanical strength of materials (see, e.g., Gordon 1976, Ewalds and Wanhill 1984). There has consequently been an enormous amount of effort put into the study of defect formation and growth, and into the study of their effect on the strength of materials. Usually these studies involve one or a few defects in an otherwise perfect lattice, as the problem of a random distribution of defects is very difficult. Recently the latter problem has been addressed using percolation models of breakdown (Takayasu 1985, de Arcangelis *et al* 1985, Sahimi and Goddard 1986, Duxbury *et al* 1986a, b). In addition to a direct simulation of the breakdown problem on percolation networks, it is possible to make an analytic study near the percolation point using standard scaling arguments (Duxbury *et al* 1986a, b) and near the pure limit by considering the most critical defects in the network (Duxbury *et al* 1986a, b). In this letter, we show by using a new distribution function scaling equation that it is possible to find not only the average strength of the networks, but also the distribution of breakdown strengths occurring in this class of model. The essential features of the analytic calculations include the cluster size and shape distributions occurring in percolation theory, the stress (voltage) enhancement occurring near the tip of the most critical defects and a new sort of scaling theory based on the stability hypothesis of the statistics of extremes.

As well as its intrinsic interest, the problem of finding the distribution function of breakdown strengths occurring in percolation models is important for the following practical reason. In many engineering situations, the breakdown distribution is assumed to be of the form introduced by Weibull (1951):

$$F(\sigma) = 1 - \exp[-cL^d(\sigma - \sigma_0)^m] \tag{1}$$

0305-4470/87/060411+05$02.50

where $F(\sigma)$ is the probability of mechanical failure in a material upon application of an external stress σ in a material of size L^d. Here c, σ_0 and m are constants that are used as parameters in fitting the experimental data. Because of its extensive use, it is of interest to determine whether it is appropriate to the percolation models that are now being introduced into the study of breakdown phenomena, and if not, whether the correct distribution is noticeably different from it.

Of course, one must take care in comparing the predictions of simple model systems with real engineering breakdown situations. For example, defect mobility, plasticity and grain boundaries are all important in the fracture of real materials and are not included at a microscopic level in this model. However, the predictions of our calculations are *qualitatively* different than currently used failure distribution functions and we expect this qualitative difference to persist in certain real situations.

In breakdown situations, it is the largest defect that often dominates in determining the strength of the material (see, e.g., Ewalds and Wanhill 1984). In a percolation model, one is then led to ask for the size of the *largest* defect for a given value of p, where a defect is a cluster of missing bonds. Define $C(n)$ as the probability that no defect of size greater than n (i.e. a defect cluster containing no more than n missing bonds) exists in a network of size L_1^d. Now consider combining N of these hypercubes together to form a hypercube of size L^d, where $N=(L/L_1)^d$. Now in the thermodynamic limit, we expect the distribution function $C(n)$ to have the same form on the L^d and L_1^d lattices and we also expect the $C(n)$ distribution functions on the L_1^d lattices to be essentially independent. The combination of these two statements gives the distribution function scaling equation:

$$[C(n)]^N \sim C(a_N n + b_N) \tag{2}$$

where a_N and b_N are scaling variables that go to a finite value in the thermodynamic limit. Equation (2) ensures that the distribution functions on the L^d and L_1^d lattices have the same form. This equation is the same as the stability hypothesis of the statistics of extremes (Gumbel 1958) and there are several forms of solution to it. The solution relevant to a particular model is dependent on the distribution of defects occurring in the model. In particular, if the distribution of defects is exponential, as is the case for the percolation problem for $p>p_c$ (Kunz and Souillard 1977), then the probability that no defect of size greater than n exists is given in the large L limit by

$$C(n) \sim \exp[-cL^d \exp(-kn)] \tag{3}$$

where c and k are arbitrary constants. In the case of an algebraic distribution of defects, as occurs at the percolation point, the probability that no defect of size greater than n exists in the L^d network is given by

$$C(n) \sim \exp(-cL^d n^{-m}) \tag{4}$$

where c and m are constants. For the calculation of the breakdown strength of a percolation network, one is interested in the probability of a defect of a particular *shape and orientation* occurring. It can be shown (Duxbury *et al* 1986b) that this does not affect the *form* of the defect distribution (3) and hence that it is correct in the breakdown networks, where $C(n)$ is now the probability that no defect of the most critical shape and orientation, of size greater than n, occurs in the L^d network. The case of breakdown in a system with an algebraic distribution of defects is more subtle and we defer discussion of the detailed behaviour of the breakdown distribution function in this case for later work. We now concentrate on the calculation of the probability of failure in percolation models of breakdown when the defect distribution is exponential (p away from p_c) and hence where the form (3) is correct for the distribution of most critical defects.

The defect of most critical shape and orientation is the defect that has the most stress enhancement at its edges. In the random fuse network (de Arcangelis *et al* 1985) and in models of brittle fracture (Sahimi and Goddard 1986), it is a linear defect in two dimensions and a penny-shaped defect in three dimensions (Duxbury *et al* 1986a, b). In models of dielectric breakdown (Takayasu 1985, Duxbury *et al* 1986a), it is a linear defect in all dimensions. For an isolated most critical defect of size n, it may be argued that the stress enhancement at its edge is given by

$$i_{\mathrm{tip}} \sim n^{1/(d-1)} \tag{5}$$

for a rigid model of brittle fracture

$$i_{\mathrm{tip}} \sim n^{1/2(d-1)} \tag{6}$$

for elastic models of fracture

$$V_{\mathrm{tip}} \sim n^{1/(d-1)} \tag{7}$$

for the random fuse network and

$$V_{\mathrm{tip}} \sim n \tag{8}$$

for the dielectric network (Duxbury *et al* 1986b). From equations (3) and (5)-(8) it is possible to find the failure distribution function for the percolation models of breakdown in many physical situations as is done below in the case of the random fuse network (a fuller exposition of this work is provided in Duxbury *et al* (1986b)). Here we concentrate on the result for the two-dimensional fuse network case, as the numerical simulations that we have done for that model allow a detailed comparison of the predicted form with the Weibull distribution. The probability of failure upon application of an external voltage V_1 for an $L \times L$ square lattice fuse network is predicted to be (inverting (8) and substituting in (3))

$$F(V_1/L) = 1 - \exp[-cL^d \exp(-kL/V_1)] \tag{9}$$

where c and k are arbitrary constants. We also wish to compare with the Weibull form, and for the fuse network this is

$$F(V_1/L) = 1 - \exp[-cL^d (V_1/L)^m] \tag{10}$$

where c and m are arbitrary constants. We have performed detailed numerical simulations for the two-dimensional random fuse network (for details of the numerical procedure see Duxbury *et al* (1986a, b), where V_1 is the breakdown voltage as defined there) and have constructed the distribution function $F(V_1/L)$ as shown in figure 1. The distribution in this figure is constructed from 1500 configurations on a 50×50 square lattice at $p = 0.90$. A direct fitting to either of the forms (9) and (10) provides an excellent fit to the numerical data, to the accuracy of the figure. However, this does not mean that the differences between the two forms are insignificant. The region of engineering interest is at the very high reliability tail of the distribution (small V_1/L in figure 1) and it is thus important to do a more detailed comparison of the data in

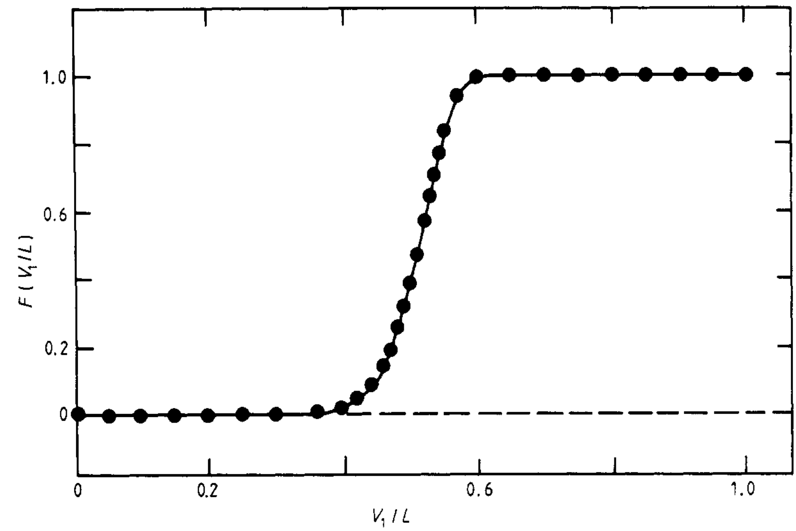

Figure 1. The probability of failure $F(V_1/L)$ constructed from 1500 realisations of the random fuse network at $p = 0.90$ on a 50×50 square lattice.

this region of the distribution function. To do this, take two logarithms of equations (9) and (10) and define

$$A = \ln\{-[\ln(1 - F(V_1/L))]/L^d\}. \tag{11}$$

For the new form predicted here, a plot of A against L/V_1 should be linear. In contrast, for the Weibull form, a plot of A against $\ln(L/V_1)$ should be linear. The data are plotted in this manner in figures 2(a) and (b) where it is seen that the new distribution function found here provides a better fit (figure 2(b)). The improvement in fit is especially noticeable for large A which is the high reliability (small V_1/L) end of the distribution displayed in figure 1.

As the percentage of defect present in real materials is not usually precisely at the percolation threshold, it appears that the form of distribution function found here should be used in fitting breakdown data rather than the Weibull form usually used. However, the models that lead to the exponential of an exponential form are very idealised and do not include many factors that may lead to a change in the form of the distribution function. Two such factors are the presence of domain boundaries and the possibility of ductility in the material. It is also apparent that the Weibull form may be recovered at the percolation threshold where a naive use of any of (6)–(8) in (4) leads to a Weibull form.

In conclusion, we have developed a new scaling equation that enables the calculation of the distribution functions of extreme defect shapes in percolation models. The use of this scaling equation combined with the stress enhancement factors for the various percolation models of breakdown enables the calculation of the failure distribution in these systems. A comparison with numerical simulations on the two-dimensional random fuse network supports the new distribution function.

We thank M Aizenmann, S Redner and J Wachtman for useful discussions and P D Beale for assistance with the numerical calculations. PMD thanks the hospitality of the Physics Department at Rutgers University where most of this work was done.

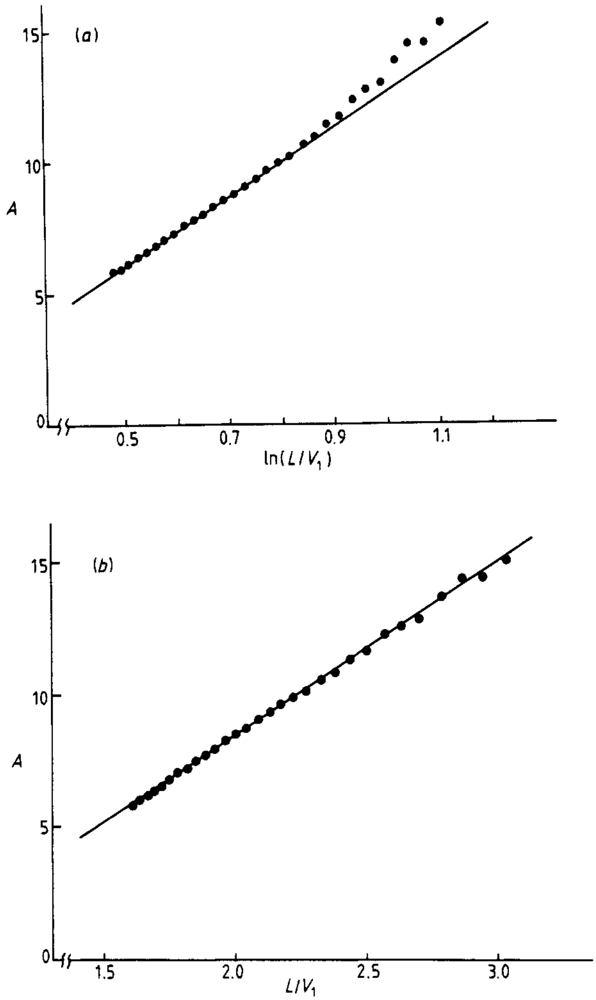

Figure 2. The data of figure 1 analysed to distinguish between the Weibull distribution (equation (10)) and the exponential of an exponential distribution (equation (9)). A is defined in equation (11) of the text and the full dots are the numerical data while the full line is an aid to the eye. (a) A against $\ln(L/V_1)$. This should be a straight line if the Weibull distribution is correct. (b) A against L/V_1. This should be a straight line if the exponential of an exponential distribution is correct.

References

de Arcangelis L, Redner S and Herrmann H J 1985 *J. Physique Lett.* **46** 585
Duxbury P M, Beale P D and Leath P L 1986a *Phys. Rev. Lett.* **57** 1052
Duxbury P M, Leath P L and Beale P D 1986b *Preprint*
Ewalds H L and Wanhill R J H 1984 *Fracture Mechanics* (London: Edward Arnold)
Gordon J E 1976 *The New Science of Strong Materials* (Princeton, NJ: Princeton University Press)
Gumbel E J 1958 *Statistics of Extremes* (New York: Columbia University Press)
Kunz H and Souillard B 1977 *Phys. Rev. Lett.* **40** 133
Sahimi M and Goddard J D 1986 *Phys. Rev.* B **33** 7848
Takayasu H 1985 *Phys. Rev. Lett.* **54** 1099
Weibull W 1951 *J. Appl. Mech.* **18** 293